Problem 1.16

The exponential atmosphere.

(a) Consider a horizontal slab of air whose thickness (height) is dz. If this slab is at rest, the pressure holding it up from below must balance both the pressure from above and the weight of the slab. Use this fact to find an expression for dP/dz, the variation of pressure with altitude, in terms of the density of air.

(b) Use the ideal gas law to write the density of air in terms of pressure, temperature, and the average mass m of the air molecules. (The information needed to calculate m is given in Problem 1.14.) Show, then, that the pressure obeys the differential equation

$$\frac{dP}{dz} = -\frac{mg}{kT}P,$$

called the **barometric equation**.

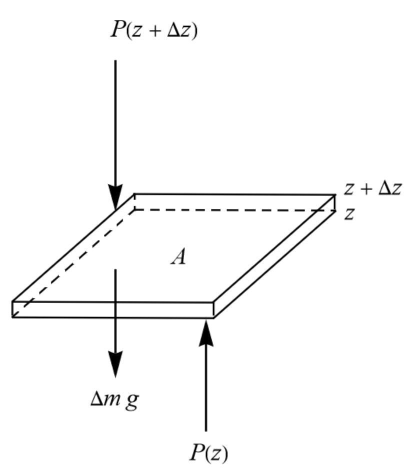
(c) Assuming that the temperature of the atmosphere is independent of height (not a great assumption but not terrible either), solve the barometric equation to obtain the pressure as a function of height: $P(z) = P(0)e^{-mgz/kT}$. Show also that the density obeys a similar equation.

(d) Estimate the pressure, in atmospheres, at the following locations: Ogden, Utah (4700 ft or 1430 m above sea level); Leadville, Colorado (10,150 ft, 3090 m); Mt. Whitney, California (14,500 ft, 4420 m); Mt. Everest, Nepal/Tibet (29,000 ft, 8850 m). (Assume that the pressure at sea level is 1 atm.)

Solution

Part (a)

Draw a free-body diagram for a horizontal slab of air whose thickness is Δz, whose mass is Δm, and whose area is A.



Use Newton's second law and consider the sum of the forces on this slab in the vertical direction.

$$\sum F_z = ma_z$$

Since the slab is at rest, the acceleration is zero. Multiply the pressures by the area to get forces.

$$P(z)A - P(z+\Delta z)A - \Delta m\, g = 0$$

The mass is obtained by multiplying the density by the volume.

$$P(z)A - P(z+\Delta z)A - (\rho \Delta V)g = 0$$

The volume is obtained by multiplying the area by the thickness.

$$P(z)A - P(z+\Delta z)A - \rho(A\Delta z)g = 0$$

$$P(z) - P(z+\Delta z) - \rho g \Delta z = 0$$

$$-\rho g \Delta z = P(z+\Delta z) - P(z)$$

Divide both sides by Δz.

$$-\rho g = \frac{P(z+\Delta z) - P(z)}{\Delta z}$$

Take the limit of both sides as $\Delta z \to 0$.

$$\lim_{\Delta z \to 0}(-\rho g) = \lim_{\Delta z \to 0} \frac{P(z+\Delta z) - P(z)}{\Delta z}$$

The right side is how the first derivative is defined.

$$-\rho g = \frac{dP}{dz} \tag{1}$$

Part (b)

Assume that the air is ideal so that the ideal gas law applies.

$$PV = nRT$$

The number of moles n is the total mass M divided by the molar mass $\mathscr{M}$.

$$PV = \left(\frac{M}{\mathscr{M}}\right) RT$$

Solve for M/V, the gas density.

$$\begin{aligned}\rho = \frac{M}{V} &= \frac{P\mathscr{M}}{RT} \\ &= \frac{P\mathscr{M}}{RT} \cdot \frac{N_A}{N_A} \\ &= \frac{P\left(\frac{\mathscr{M}}{N_A}\right)}{T}\left(\frac{N_A}{R}\right) \\ &= \frac{Pm}{T}\left(\frac{1}{k}\right) \\ &= \frac{Pm}{kT} \qquad (2)\end{aligned}$$

m is the mass (in kilograms) per gas molecule. Substituting this formula into equation (1),

$$\frac{dP}{dz} = -\rho g = -\left(\frac{Pm}{kT}\right) g.$$

Therefore,

$$\frac{dP}{dz} - \frac{mg}{kT} P.$$

Part (c)

If the temperature is independent of height, then T is a constant in this ODE. Solve it by separating variables.

$$\frac{dP}{P} = -\frac{mg}{kT}\, dz$$

Integrate both sides.

$$\int \frac{dP}{P} = \int -\frac{mg}{kT}\, dz$$

$$\ln |P| = -\frac{mg}{kT} z + C$$

Exponentiate both sides to solve for P.

$$\begin{aligned} |P| &= \exp\left(-\frac{mg}{kT} z + C\right) \\ &= e^C \exp\left(-\frac{mg}{kT} z\right) \end{aligned}$$

Place $\pm$ on the right side to remove the absolute value sign.

$$P(z) = \pm e^C \exp\left(-\frac{mg}{kT} z\right)$$

Use a new constant A for $\pm e^C$.

$$P(z) = A \exp\left(-\frac{mg}{kT} z\right)$$

This equation holds for every value of z, so A can be determined by setting $z = 0$.

$$P(0) = A e^0 = A$$

Therefore, the pressure as a function of height is

$$\boxed{P(z) = P(0) \exp\left(-\frac{mg}{kT} z\right).}$$

The density is obtained by using equation (2).

$$\begin{aligned} \rho(z) &= \frac{P(z)m}{kT} \\ &= \frac{P(0)m}{kT} \exp\left(-\frac{mg}{kT} z\right) \end{aligned}$$

Part (d)

Sea level is at $z = 0$, so $P(0) = 1$ atm. Assuming the air is dry air and doesn't vary in composition with height, the molar mass is $\mathscr{M} = 28.97$ g/mol, which means

$$\begin{aligned} m &= \frac{\mathscr{M}}{N_A} \\ &= 28.97\ \frac{\cancel{\text{g}}}{\cancel{\text{mol}}} \times \frac{1\ \cancel{\text{mol}}}{6.02 \times 10^{23}\ \text{molecules}} \times \frac{1\ \text{kg}}{1000\ \cancel{\text{g}}} \\ &\approx 4.81 \times 10^{-26}\ \text{kg}. \end{aligned}$$

Assume the temperature is 25°C so that $T = 298.15$ K. Therefore, at the different locations,

$$\begin{aligned} \text{Ogden, Utah } (z = 1430\text{ m}): &\quad P(1430\text{ m}) = (1\text{ atm}) \exp\left[-\frac{(4.81\times 10^{-26})(9.81)}{(1.381\times 10^{-23})(298.15)} \cdot 1430\right] \approx 0.849\text{ atm} \\ \text{Leadville, Colorado } (z = 3090\text{ m}): &\quad P(3090\text{ m}) = (1\text{ atm}) \exp\left[-\frac{(4.81\times 10^{-26})(9.81)}{(1.381\times 10^{-23})(298.15)} \cdot 3090\right] \approx 0.702\text{ atm} \\ \text{Mt. Whitney, California } (z = 4420\text{ m}): &\quad P(4420\text{ m}) = (1\text{ atm}) \exp\left[-\frac{(4.81\times 10^{-26})(9.81)}{(1.381\times 10^{-23})(298.15)} \cdot 4420\right] \approx 0.602\text{ atm} \\ \text{Mt. Everest, Nepal/Tibet } (z = 8850\text{ m}): &\quad P(8850\text{ m}) = (1\text{ atm}) \exp\left[-\frac{(4.81\times 10^{-26})(9.81)}{(1.381\times 10^{-23})(298.15)} \cdot 8850\right] \approx 0.363\text{ atm}. \end{aligned}$$